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Adhesively bonded assemblies with identical nondeformable adherends and 'piecewise continuous' adhesive layer: predicted thermal stresses in the adhesive

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Abstract

We consider a thermoelastic problem for an elongated adhesively bonded assembly with identical nondeformable adherends and a 'piecewise continuous' adhesive layer: the adhesive layer consists of a large number of 'pieces' that differ by their lengths, Young's moduli, Poisson's ratios, and coefficients of thermal expansion. Assemblies of this type are of interest in connection with the manufacturing, and mechanical and optical performance of some photonics structures. We develop a stress analysis model for the evaluation of thermally induced stresses, strains and displacements in the adhesive layer. These stresses are due to the thermal expansion (contraction) mismatch of the adhesive material with the material of the adherends, as well as to the mismatch between the adjacent 'pieces' of the adhesive layer.

1. Introduction

Adhesively bonded assemblies, subjected to mechanical or thermal loading, are widely used in engineering. The mechanical behavior of such assemblies was analyzed in numerous studies (see, for instance, Völkerson, 1938; de Bruyne, 1944; Goland and Reissner, 1944; Hart-Smith, 1973a, b, c; Suhir, 1986, 1994, 1997; Lin and Lin, 1993; Tsai and Morton, 1995). Adhesively bonded assemblies are typically manufactured at an elevated (curing) temperature and subsequently cooled down to a low (room, testing, or operation) temperature. If the adherends are made of dissimilar materials, thermally induced stresses, caused by the thermal contraction mismatch of these materials, arise at low temperature conditions.

There is an obvious incentive to employ identical adherends for lower interfacial stresses in, and a

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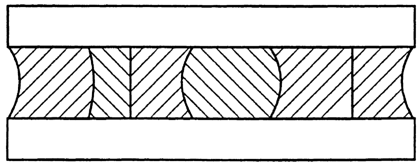

Fig. 1. An adhesively bonded assembly with a 'piecewise-continuous' adhesive layer.

lower bow of, the adherends. It is clear that in such a case the assembly, as a whole, will not experience bending, but each of the adherends will bow, to a greater or lesser extent, with respect to the midplane of the assembly. It is clear also that the thermal expansion (contraction) of the adhesive, which usually need not be considered for dissimilar adherends (as long as the adhesive layer is thin and the adhesive material has low Young's modulus), plays an important role in the case of thermally matched adherends (see, for instance, Suhir, 1989).

In the analysis which follows we consider a thermoelastic problem for an elongated adhesively bonded assembly with identical adherends and a 'piecewise continuous' adhesive layer: this layer consists of a large number of 'pieces' that might differ by their lengths, Young's moduli, Poisson's ratios, and coefficients of expansion (Fig. 1). The bow of the adherends is considered small enough, so that they can be treated as nondeformable ones. Assemblies of this type are of interest, for instance, in connection with their application in some photonics structures.

We examine a situation when the assembly is manufactured at an elevated temperature and is then cooled down to a low temperature. The coefficient of thermal expansion (contraction) of the adhesive material is assumed to be considerably larger than that of the adherends.

The objective of the analysis is to develop an easy-to-use stress model for the evaluation of thermally induced stresses, strains and displacements in the adhesive layer. These stresses and deformations are due to the thermal contraction mismatch of the adhesive material with the material of the adherends, as well as to the mismatch of the adjacent 'pieces' of the adhesive layer with each other. The developed stress model is intended to be used primarily for the evaluation of the effect of the thickness of the adhesive layer, the lengths of its 'pieces', and the mechanical properties of the employed materials on the stresses and displacements in the adhesive material. Particularly, we intend to find out whether the adhesive material can be chosen and the adhesive layer can be designed in such a way that the boundaries of the particular 'pieces' of this layer remain straight (undistorted) despite the temperature excursions, i.e. will not experience longitudinal (inplane) displacements as a result of temperature change. The analysis is based on an elementary Structural Analysis (Strength of Materials) approach, rather than on a Theory-of-Elasticity treatment of the problem.

The interfacial stresses and the adherends' bow are addressed in Appendix A. The results of the analysis enable one to establish the conditions under which the adherends could be indeed treated as nondeformable ones, whether a mechanical or any other criterion (say, optical) is applied.

Analysis of stresses and displacements in the adhesive layer of adhesively bonded assemblies with identical nondeformable adherends, carried out for an elongated (long-and-narrow) and a circular assembly (Suhir, 1999), was then extended for the case of a nonhomogeneous adhesive layer (Suhir, 1997), whose mechanical properties in its midportion were different from those in the peripheral areas. The analysis, which is set forth below, is in effect, a generalization of the developed stress model for the case of a large number of nonhomogeneous portions ('pieces').

1.1. Basic equation: 'Theorem of three boundary forces'

Let an elongated adhesively bonded assembly with identical nondeformable adherends and a

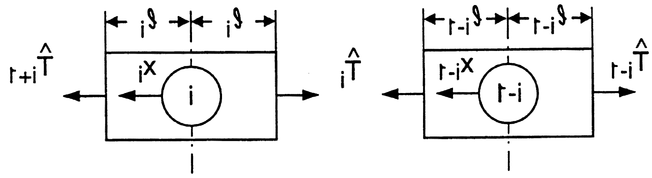

Fig. 2. Boundary forces acting between the adhesive 'pieces'.

'piecewise continuous' adhesive layer (Fig. 1) be manufactured at an elevated temperature and subsequently cooled down to a lower temperature. This results in interfacial shearing stresses, $\tau_i(x_i)$, acting in the i-th 'piece' of the adhesive layer, and in boundary forces, $\hat{T}_i$, acting between the (i–1)-st and i-th 'pieces' of this layer (Fig. 2). The displacement, $u_i(-l_i)$, at the edge $x_i = -l_i$ of the i-th 'piece' can be expressed, using the formulas (B5) of Appendix B, as

$$u_i(-l_i) = \kappa_i\left(\frac{\Delta\alpha_i\Delta_t}{k_i\kappa_i}\tanh k_il_i - k_i\hat{T}_i \operatorname{cotanh} 2k_il_i + \frac{k_i\hat{T}_{i+1}}{\sinh 2k_il_i}\right). \tag{1}$$

Here κ_i is the interfacial compliance of the i-th 'piece' of the adhesive layer, λ_i is its longitudinal compliance, l_i is half the 'piece' length, $\Delta\alpha_i$ is the difference between the coefficients of thermal expansion of the material of the given 'piece' of the adhesive layer and the material of the 'adherends', and Δt is the change in temperature.

The first term in the parentheses in formula (1) is the interfacial shearing stress caused by the thermal contraction mismatch of the i-th 'piece' of the adhesive layer with the material of the adherends (Suhir, 1986). The second term is the interfacial shearing stress caused by the force, $\hat{T}_i$, applied at the i-th boundary, i.e. at the boundary at which the displacement, $u_i(-l_i)$, is sought (see Appendix B). The third term is the interfacial shearing stress at the i-th boundary due to the force, $\hat{T}_{i+1}$, applied at the (i + 1)-st boundary (see Appendix B). The forces, $\hat{T}$, are considered positive, if they are tensile ones. The second term in (1) is taken with a sign 'minus' since the boundary force, $\hat{T}_i$, reduces the displacement at the i-th boundary.

The formula for the displacement of the ($i-1$)-st 'piece' of the adhesive layer at the edge $x_{i-1} = l_{i-1}$ can be written in a similar fashion as

$$u_{i-1}(l_{i-1}) = \kappa_{i-1}\left(-\frac{\Delta\alpha_{i-1}\Delta_t}{k_{i-1}\kappa_{i-1}}\tanh k_{i-1}l_{i-1} + k_{i-1}\hat{T}_i \operatorname{cotanh} 2k_{i-1}l_{i-1} - \frac{k_{i-1}\hat{T}_{i-1}}{\sinh 2k_{i-1}l_{i-1}}\right). \tag{2}$$

The condition of the compatibility of displacements requires that the boundary displacements, $u_i(-l_i)$ and $u_{i-1}(l_{i-1})$, expressed by formulas (1) and (2), respectively, be equal. This results in the following equation for the unknown boundary forces, $\hat{T}_{i-1}$, $\hat{T}_i$, and $\hat{T}_{i+1}$:

$$-\delta_{i-1}\hat{T}_{i-1} + \delta_i\hat{T}_i - \delta_{i+1}\hat{T}_{i+1} = \Delta_i, \quad i = 1, 2, \ldots \tag{3}$$

where

$$\delta_{i-1} = \frac{k_i\lambda_{i-1}}{\sinh 2k_{i-1}l_{i-1}},$$

$$\delta_i = k_i\lambda_{i-1} \operatorname{cotanh} 2k_{i-1}l_{i-1} + k_{i-1}\lambda_i \operatorname{cotanh} 2k_il_i = \delta_{i-1}\cosh 2k_{i-1}l_{i-1} + \delta_{i+1}\cosh 2k_il_i,$$

$$\delta_{i+1} = \frac{k_{i-1}\lambda_i}{\sinh 2k_i l_i} \tag{4}$$

and the 'load term', Δ_i, is expressed as

$$\Delta_i = (k_i \Delta\alpha_{i-1} \tanh k_{i-1} l_{i-1} + k_{i-1}\Delta\alpha_i \tanh k_i l_i)\Delta t, \quad i = 1, 2, \ldots \tag{5}$$

The obtained equations are analogous to the equations, expressed by the well-known 'Theorem of Three Moments' in the theory of continuous beams with multiple supports (see, for instance, Timoshenko, 1956 or Suhir, 1991), and therefore the corresponding stress model can be called 'Theorem of Three Boundary Forces'.

1.2. Special cases

1.2.1. 'Single piece' adhesive layer

In this case (Suhir, 1999), one can put, when examining the *i*-th 'piece' of the adhesive layer, $\hat{T}_{i-1} = \hat{T}_{i+1} = 0$, $\lambda_{i-1} = 0$, and $\Delta\alpha_{i-1} = 0$. Then eqn (3) yields:

$$\hat{T}_i = \frac{\Delta\alpha_i \Delta t}{\lambda_i} \tanh k_i l_i \tanh 2k_i l_i = E_i^0 k_0 \Delta\alpha_i \Delta t \tanh k_i l_i \tanh 2k_i l_i, \tag{6}$$

where $E_i^0 = [E_i/(1-v_i)]$ is effective Young's modulus of the adhesive material.

When the *i*-th 'piece' is long (when the product $k_i l_i$ is larger than, say, 2.5), one can put $\tanh k_i l_i \simeq \tanh 2k_i l \simeq 1$, and formula (6) yields:

$$\hat{T}_i = E_i^0 h_0 \Delta\alpha_i \Delta t. \tag{7}$$

The normal stress can be found in this case as

$$\sigma_i = \frac{\hat{T}_i}{h_0} = E_i^0 \Delta\alpha_i \Delta t. \tag{8}$$

The requirement $k_i l_i > 2.5$ is equivalent to the condition:

$$2l_i > 5.0\sqrt{\frac{\kappa_i}{\lambda_i}}. \tag{9}$$

The formula (B3) of the Appendix B, in the case of a long 'piece' of the adhesive layer, can be simplified, and results in the following formula for the interfacial compliance (Suhir, 1986):

$$\kappa_i = \frac{1+v_i}{3E_i} h_0. \tag{10}$$

Note that this compliance depends on the thickness of the *i*-th 'piece' of the adhesive layer and is independent of the length of this 'piece'.

Considering formula (B2) of Appendix B, one can write condition (9) as

$$\frac{2l_i}{h_0} > 5\sqrt{\frac{1}{3}\frac{1+v_i}{1-v_i}}. \tag{11}$$

Young's modulus of the adhesive material and its coefficient-of-expansion do not enter this formula.

Hence, these properties do not affect the conclusion whether the given 'piece' of the adhesive layer should or should not be treated as an infinitely long one. Only Poisson's ratio is important.

The right part of the condition (11) is the largest, when Poisson's ratio is $v_i = 0.5$ (noncompressible adhesive material). This yields:

$$\frac{2l_i}{h_0} > 5. \tag{12}$$

Thus, if the length of the given 'piece' of the adhesive layer is at least five-fold larger than its thickness, this 'piece' can be treated, as far as the boundary forces are concerned, as an infinitely long one.

In the opposite extreme case, when the i-th 'piece' of the adhesive layer is very short (say, $k_i l_i < 0.4$), formula (6) yields:

$$\hat{T}_i = 2\frac{\Delta\alpha_i \Delta t}{\kappa_i} l_i. \tag{13}$$

For low $k_i l_i$ values, the expression (B3) of the Appendix B results in the following simple formula for the interfacial compliance (Suhir, 1986):

$$\kappa_i = \frac{(1 + v_i)(3 - v_i)}{3E_i} l_i. \tag{14}$$

In this case, the compliance depends on the length of the adhesive 'piece' and is independent of the thickness of the adhesive layer. With the expression (14) for the compliance, formula (13) yields:

$$\hat{T}_i = 6E_i^0 \frac{(1 - v_i) l_i}{(1 + v_i)(3 - v_i)} \Delta\alpha_i \Delta t, \tag{15}$$

and the corresponding stress is

$$\sigma_i = 6E_i^0 \frac{1 - v_i}{(1 + v_i)(3 - v_i)} \frac{l_i}{h_0} \Delta\alpha_i \Delta t. \tag{16}$$

The requirement $k_i l_i < 0.4$ is equivalent to the condition

$$\frac{2l_i}{h} < \frac{8}{75} \frac{(1 + v_i)(3 - v_i)}{1 - v_i}. \tag{17}$$

The right part of this condition is the lowest for the lowest possible v_i value. Actual materials are typically characterized by Poisson's ratios that are not lower than 0.2. For $v_i = 0.2$, the condition (17) yields:

$$\frac{2l_i}{h_0} < 0.448. \tag{18}$$

Note that for $v_i = 0.5$ this requirement would be considerably less stringent:

$$\frac{2l_i}{h_0} < 0.800. \tag{19}$$

1.2.2. 'Three-piece' ('bi-material') adhesive layer: the peripheral portions of the adhesive layer are identical but are different from its midportion

In this case one can put $i = 1$, $T_0 = 0$, $T_2 = T_1$, $\lambda_0 = \lambda_2$, $k_0 = k_2$, and $l_0 = l_2$, and formula (3) yields (Suhir, 1998):

$$\hat{T}_1 = \frac{k_2 \Delta\alpha_1 \tanh k_1 l_1 + k_1 \Delta\alpha_2 \tanh k_2 l_2}{k_2 \lambda_1 \operatorname{cotanh} 2k_1 l_1 + k_1 \lambda_2 \operatorname{cotanh} 2k_2 l_2 - \dfrac{k_2 \lambda_1}{\sinh 2k_1 l_1}} \Delta t. \tag{20}$$

If all three 'pieces' of the adhesive layer are sufficiently long, formula (20) can be simplified as follows:

$$\hat{T}_1 = \frac{k_2 \Delta\alpha_1 + k_1 \Delta\alpha_2}{k_1 \lambda_2 + k_2 \lambda_1} \Delta t. \tag{21}$$

When $k_1 = k_2 = k$, $\lambda_1 = \lambda_2 = \lambda$, and $\Delta\alpha_1 = \Delta\alpha_2 = \Delta\alpha$, this formula leads to formula (7).

1.3. Zero distortion conditions

The conditions of zero distortion of the boundaries of the 'pieces' of the adhesive layer can be written, using formulas (1) and (2), as follows:

$$\begin{aligned} &\hat{T}_i \operatorname{cotanh} 2k_i l_i - \frac{\hat{T}_{i+1}}{\sinh 2k_i l_i} = \frac{\Delta\alpha_i \Delta t}{\lambda_i} \tanh k_i l_i \\ &- \frac{\hat{T}_{i-1}}{\sinh 2k_{i-1} l_{i-1}} + \hat{T}_i \operatorname{cotanh} 2k_{i-1} l_{i-1} = \frac{\Delta\alpha_{i-1} \Delta t}{\lambda_{i-1}} \tanh k_{i-1} l_{i-1}. \end{aligned} \tag{22}$$

These conditions, taken together, are 'stronger' than the requirement underlying eqn (3). Indeed, this equation requires that the boundary displacements, whatever their magnitudes, are the same, while conditions (22) impose an additional requirement that all these displacements be zero. As a matter of fact, the 'Three Boundary Force Equation' (3) can be obtained also by summing up eqns (22), i.e. by substituting the two eqns (22) with a single eqn (3).

If one requires that the boundary forces are the same at all the boundaries, i.e. that $\hat{T}_{i+1} = \hat{T}_i = \hat{T}_{i-1}$, then the eqns (22) yield:

$$\hat{T}_i = T_i^{\infty} = \frac{\Delta\alpha_i \Delta t}{\lambda_i}, \quad i = 1, 2, \ldots \tag{23}$$

This can take place only in a situation, when all the 'pieces' of the adhesive layer are sufficiently long. Indeed, formula (23) can be obtained from eqns 22 also by putting $k_{i-1} l_{i-1} \to \infty$ and $k_i l_i \to \infty$.

The formula (23) can be used to formulate the requirement for the materials characteristics that would lead to distortion-free boundaries in an adhesive layer with sufficiently long 'pieces'. This formula indicates that the boundaries of such 'pieces' will remain undistorted, if the product $E_i^0 \Delta\alpha_i \Delta t$ is kept constant throughout the adhesive layer, i.e. if effective Young's moduli of any two 'pieces', i and j, of this layer are inversely proportional to the thermal expansion (contraction) mismatch strains of the materials of these 'pieces' with the material of the adherends:

$$\frac{E_i^0}{E_j^0} = \frac{\Delta\alpha_j \Delta t}{\Delta\alpha_i \Delta t}. \tag{24}$$

Table 1
Materials' properties and calculated boundary distortions (displacements), u_i

Boundary, i	0	1	2	3	4	5	6
Young's modulus, E_i, kg/mm^2	0.141	0.282	0.423	0.564	0.423	0.282	—
Poisson's ratio, ν_i	0.50	0.49	0.48	0.47	0.48	0.49	—
Effective Young's modulus, $E_i^0 = [E_i/(1-\nu_i)]$, kg/mm^2	0.282	0.553	0.813	1.064	0.813	0.553	—
Thermally induced strain, $\varepsilon_i = \Delta\alpha_i \Delta t$	0.02	0.04	0.06	0.08	0.06	0.04	—
'Piece' length, $2l_i$, mm	1.25	1.25	1.25	1.25	1.25	1.25	—
Axial compliance of the adhesive layer, $(\lambda_0)_i = (1/E_i^0 h_i)$, mm/kg	14.184	7.233	4.920	3.759	4.920	7.233	—
Shearing compliance of the adhesive layer, $(\kappa_0)i = (1+\nu_i/3E_i)h_0$, mm^3/kg	0.8865	0.4403	0.2916	0.2172	0.2916	0.4403	—
Eigenvalue of the problem, $k_i = \sqrt{(\lambda_0)_i/(\kappa_0)_i}$, mm^{-1}	4.000	4.053	4.108	4.160	4.108	4.053	—
Length parameter, $k_i l_i$	2.500	2.533	2.567	2.600	2.567	2.533	—
sinh $(2k_i l_i)$	74.203	79.2663	84.844	90.633	84.844	79.266	—
cotanh $(2k_i l_i)$	1	1	1	1	1	1	—
tanh $(k_i l_i)$	0.9866	0.9875	0.9883	0.9890	0.9883	0.9875	—
ε_i tanh $k_i l_i$	0.01973	0.03950	0.05930	0.07912	0.05930	0.03950	—
Boundary force per unit assembly width, $\hat{T}_i$, g/mm	0	2.788	8.204	16.052	16.050	8.184	—
$E_i^0 \varepsilon_i$ tanh $(k_i l_i)$, g/mm^2	5.564	21.843	48.211	84.184	48.211	21.843	—
Shearing stress, $(\tau_0)_i$, g/mm^2	0.150	−10.880	−32.925	−66.040	−65.537	−33.170	−0.418
Boundary distortion, u_i, μm	0.133	−4.791	−9.601	−14.344	−19.111	−14.605	−0.184

2. Numerical example

Let a 0.25 mm thick adhesive layer of an adhesively bonded assembly with identical nondeformable adherends be comprised of five 1.25 mm long $[(l/h_0) = 2.5]$ 'pieces'. The mechanical characteristics of these 'pieces' are given in Table 1. The calculations of the boundary forces, based on eqns (3), are carried out in Table 2. The calculated forces are shown in Table 1 in the fourth line from the bottom. These forces are the highest in the midportion of the adhesive layer, in which generalized Young's moduli, E_i^0, and the induced strains, ε_i, are the largest.

The shearing stress, τ_i, in the i-th 'piece' at its i-th (left) boundary, i.e. at the boundary of this 'piece' with the $(i-1)$-st 'piece' of the adhesive layer, can be calculated, based on the formulas (B5) of the Appendix B, as follows:

$$\tau_i = -k_i\left(\hat{T}_i \operatorname{cotanh} 2k_i l_i - \frac{\hat{T}_{i+1}}{\sinh 2k_i l_i}\right).$$

The corresponding boundary displacement is $u_i = \kappa_i \tau_i$. Using the input information and the calculated boundary forces from Table 1, we obtain:

$$\tau_0 = k_0 \frac{\hat{T}_1}{\sinh 2k_0 l_0} = \frac{4.000 \times 2.788}{74.2032} = 0.15029 \text{ g/mm}^2,$$

Table 2
Calculated boundary forces, $\hat{T}_i$

$\underline{i=1}$; $\delta_1\hat{T}_1 - \delta_2\hat{T}_2 = \Delta_1$;
$\delta_1 = k_1\lambda_0 \text{ cotanh } 2k_0l_0 + k_1\lambda_1 \text{ cotanh } 2k_1l_1$
$= 4.053 \times 14.184 + 4.000 \times 7.233 = 86.420 \text{ kg}^{-1}$;
$\delta_2 = \dfrac{k_0\lambda_1}{\sinh 2k_1l_1} = \dfrac{4.000 \times 7.233}{79.2663} = 0.3650 \text{ kg}^{-1}$;
$\Delta_1 = k_1\varepsilon_0 \tanh k_0l_0 + k_0\varepsilon_1 \tanh k_1l_1$
$= 4.053 \times 0.01973 + 4.000 \times 0.03950 = 0.2380 \text{ mm}^{-1}$;
$86.420\hat{T}_1 - 0.365\hat{T}_2 = 0.238; 236.767\hat{T}_1 - \hat{T}_2 = 0.6520 \quad (1)$

$\underline{i=2}$; $-\delta_1\hat{T}_1 + \delta_2\hat{T}_2 - \delta_3\hat{T}_3 = \Delta_2$;
$\delta_1 = \dfrac{k_2\lambda_1}{\sinh 2k_1l_1} = \dfrac{4.108 \times 7.233}{79.2663} = 0.3748 \text{ kg}^{-1}$;
$\delta_2 = k_2\lambda_1 \text{ cotanh } 2k_1l_1 + k_1\lambda_2 \text{ cotanh } 2k_2l_2$
$= 4.108 \times 7.233 + 4.053 \times 4.920 = 49.654 \text{ kg}^{-1}$;
$\delta_3 = \dfrac{k_1\lambda_1}{\sinh 2k_2l_2} = \dfrac{4.053 \times 4.920}{84.8443} = 0.2350 \text{ kg}^{-1}$;
$\Delta_2 = k_2\varepsilon_1 \tanh k_1l_1 + k_1\varepsilon_2 \tanh k_2l_2$
$= 4.108 \times 0.03950 + 4.053 \times 0.05930 = 0.4026 \text{ mm}^{-1}$;
$-0.3748\hat{T}_1 + 49.654\hat{T}_2 - 0.2350\hat{T}_3 = 0.4026; -1.595\hat{T}_1 + 211.294\hat{T}_2 - \hat{T}_3 = 1.713 \quad (2)$

$\underline{i=3}$; $-\delta_2\hat{T}_2 + \delta_3\hat{T}_3 - \delta_4\hat{T}_4 = \Delta_3$;
$\delta_2 = \dfrac{k_3\lambda_2}{\sinh 2k_2l_2} = \dfrac{4.160 \times 4.920}{84.8443} = 0.2412 \text{ kg}^{-1}$;
$\delta_3 = k_3\lambda_2 \text{ cotanh } 2k_2l_2 + k_2\lambda_3 \text{ cotanh } 2k_3l_3$
$= 4.160 \times 4.920 + 4.108 \times 3.759 = 35.909 \text{ kg}^{-1}$;
$\delta_4 = \dfrac{k_2\lambda_3}{\sinh 2k_3l_3} = \dfrac{4.108 \times 3.759}{90.6334} = 0.1704 \text{ kg}^{-1}$;
$\Delta_3 = k_3\varepsilon_2 \tanh k_2l_2 + k_2\varepsilon_3 \tanh k_3l_3$
$= 4.160 \times 0.0593 + 4.108 \times 0.07912 = 0.5717 \text{ mm}^{-1}$;
$-0.2412\hat{T}_2 + 35.909\hat{T}_3 - 0.1704\hat{T}_4 = 0.5717$;
$-1.415\hat{T}_2 + 210.733\hat{T}_3 - \hat{T}_4 = 3.355 \quad (3)$

$\underline{i=4}$; $-\delta_3\hat{T}_3 + \delta_4\hat{T}_4 - \delta_5\hat{T}_5 = \Delta_4$;
$\delta_3 = \dfrac{k_4\lambda_3}{\sinh 2k_3l_3} = \dfrac{4.108 \times 3.759}{90.6334} = 0.1704 \text{ kg}^{-1}$;
$\delta_4 = k_4\lambda_3 \text{ cotanh } 2k_3l_3 + k_3\lambda_4 \text{ cotanh } 2k_4l_4$
$= 4.108 \times 3.759 + 4.160 \times 4.920 = 35.909 \text{ kg}^{-1}$;
$\delta_5 = \dfrac{k_3\lambda_4}{\sinh 2k_4l_4} = \dfrac{4.160 \times 4.920}{84.8443} = 0.2412 \text{ kg}^{-1}$;
$\Delta_4 = k_4\varepsilon_3 \tanh k_3l_3 + k_3\varepsilon_4 \tanh k_4l_4$
$= 4.108 \times 0.07912 + 4.160 \times 0.0593 = 0.5717 \text{ mm}^{-1}$;
$-0.1704\hat{T}_3 + 35.909\hat{T}_4 - 0.2412\hat{T}_5 = 0.5717$
$-0.706\hat{T}_3 + 148.876\hat{T}_4 - \hat{T}_5 = 2.370 \quad (4)$

$\underline{i=5}$; $-\delta_4\hat{T}_4 + \delta_5\hat{T}_5 = \Delta_5$;
$\delta_4 = \dfrac{k_5\lambda_4}{\sinh 2k_4l_4} = \dfrac{4.053 \times 4.920}{84.8443} = 0.2350 \text{ kg}^{-1}$;
$\delta_5 = k_5\lambda_4 \text{ cotanh } 2k_4l_4 + k_4\lambda_5 \text{ cotanh } 2k_5l_5$
$= 4.053 \times 4.920 + 4.108 \times 7.233 = 49.654 \text{ kg}^{-1}$;

$\delta_5 = k_5\varepsilon_4 \tanh k_4l_4 + k_4\varepsilon_5 \tanh k_5l_5$
$= 4.053 \times 0.0593 + 4.108 \times 0.0395 = 0.4026 \text{ mm}^{-1}$;
$-0.235\hat{T}_4 + 49.654\hat{T}_5 = 0.4026$
$-0.004732\hat{T}_4 + \hat{T}_5 = 0.008108 \quad (5)$

Table 2 (*continued*)

The equations (4) and (5) result in the relationship:
$-0.706\hat{T}_3 + 148.872\hat{T}_4 = 2.3780 - 0.004742\hat{T}_3 + \hat{T}_4 = 0.01597;$ (6)

From (3) and (6) we have:
$-1.415\hat{T}_2 + 210.728\hat{T}_3 = 3.371 - 0.006715\hat{T}_2 + \hat{T}_3 = 0.01600;$ (7)

From (2) and (7) we find:
$-1.595\hat{T}_1 + 211.287\hat{T}_2 = 1.7290 - 0.007549\hat{T}_1 + \hat{T}_2 = 0.008183;$ (8)

From (1) and (8) we obtain:
$236.759\hat{T}_1 = 0.66018;\ \hat{T}_1 = 0.002788$ kg/mm = 2.788 g/mm;

Then the forces $\hat{T}_2$, $\hat{T}_3$, $\hat{T}_4$ and $\hat{T}_5$ can be computed from the eqns (8), (7), (6) and (5), respectively:
$\hat{T}_2 = 0.008204$ kg/mm = 8.204 g/mm; $\hat{T}_3 = 0.016052$ kg/mm = 16.052 g/mm;
$\hat{T}_4 = 0.016050$ kg/mm = 16.050 g/mm; $\hat{T}_5 = 0.008184$ kg/mm = 8.184 g/mm.

$$u_0 = \kappa_0\tau_0 = 0.8865 \times 0.15029 = 0.1332\ \mu\text{m};$$

$$\tau_1 = -k_1\left(\hat{T}_1 \text{ cotanh } 2k_1l_1 - \frac{\hat{T}_2}{\sinh 2k_1l_1}\right) = -4.053\left(2.788 - \frac{8.204}{79.2663}\right) = 10.8803\ \text{g/mm}^2,$$

$$u_1 = \kappa_1\tau_1 = 0.4403 \times (-10.8803) = -4.7906\ \mu\text{m};$$

$$\tau_2 = -k_2\left(\hat{T}_2 \text{ cotanh } 2k_2l_2 - \frac{\hat{T}_3}{\sinh 2k_2l_2}\right) = -4.108\left(8.204 - \frac{16.052}{84.8443}\right) = -32.9248\ \text{g/mm}^2,$$

$$u_2 = \kappa_2\tau_2 = 0.2916 \times (-32.9248) = -9.6009\ \mu\text{m};$$

$$\tau_3 = -k_3\left(\hat{T}_3 \text{ cotanh } 2k_3l_3 - \frac{\hat{T}_4}{\sinh 2k_3l_3}\right) = -4.160\left(16.052 - \frac{16.050}{90.6334}\right) = -66.0396\ \text{g/mm}^2,$$

$$u_3 = \kappa_3\tau_3 = 0.2172 \times (-66.0396) = -14.3438\ \mu\text{m};$$

$$\tau_4 = -k_4\left(\hat{T}_4 \text{ cotanh } 2k_4l_4 - \frac{\hat{T}_5}{\sinh 2k_4l_4}\right) = -4.108\left(16.050 - \frac{8.184}{84.8443}\right) = -65.5371\ \text{g/mm}^2,$$

$$u_4 = \kappa_4\tau_4 = 0.2916 \times 65.4363 = -19.111\ \mu\text{m};$$

$$\tau_5 = -k_5\left(\hat{T}_5 \text{ cotanh } 2k_5l_5\right) = -4.053 \times 8.184 = -33.1698\ \text{g/mm}^2,$$

$$u_5 = \kappa_5\tau_5 = -0.4403 \times 33.1698 = -14.6046\ \mu\text{m};$$

$$\tau_6 = -k_5\frac{\hat{T}_5}{\sinh 2k_5l_5} = -4.053\frac{8.184}{79.2663} = -0.41846\ \text{g/mm}^2,$$

$$u_6 = \kappa_5\tau_6 = -0.4403 \times 0.41846 = -0.1842\ \mu\text{m}.$$

The predicted distortions of the boundaries of the adhesive 'pieces' are shown in Fig. 3. Note that the change in the factors $E_i^0\Delta\alpha_i\Delta t \tanh k_il_i$ can be used, as follows from eqns (22) and (23), as a suitable measure of the degree of the distortion of the boundaries. The calculated values of these factors are shown in Table 1.

3. Conclusion

A simple stress model ('Theorem of Three Boundary Forces') is developed for the evaluation of the boundary forces and the boundary displacements in an adhesively bonded assembly with identical nondeformable adherends and a 'piecewise continuous' adhesive layer. It is shown that each 'piece' of the adhesive layer can be treated, from the standpoint of the induced boundary forces and displacements, as an infinitely long one, if its actual length is at least five times larger than the layer thickness. It is shown also that in such a case the inner boundaries of the 'pieces' of the adhesive layer will remain straight (undistorted), i.e. will not be affected by the change in temperature, if the product of the effective Young's modulus of the adhesive material and the thermally induced strain in it is kept the same for all the adhesive 'pieces', i.e. if this product is kept constant throughout the assembly.

Acknowledgements

The author acknowledges, with thanks, useful discussions with Dr. S. Patel.

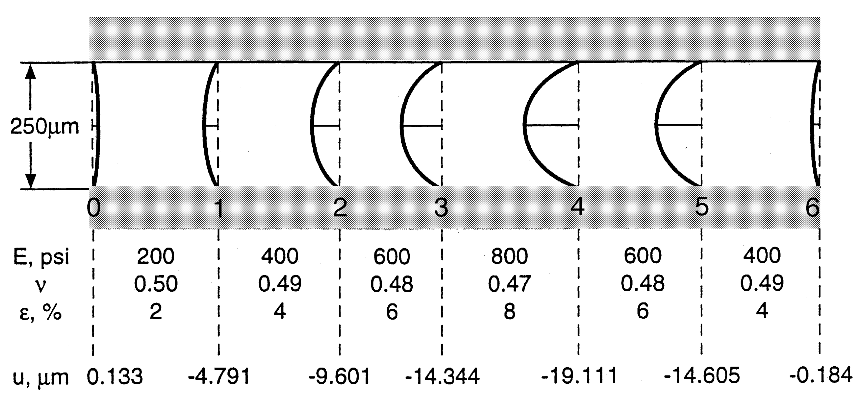

Fig. 3. Distorted boundaries of the adhesive layer.

Appendix A. Thermally induced stresses and bow of the adherends in an elongated adhesively bonded assembly with identical adherends

Basic equations

The objective of the analysis carried out in this Appendix is to develop a simple stress model for the prediction of the interfacial stresses in, and the bow of, the adherends in an elongated bimaterial adhesively bonded assembly with identical adherends. The results of this anlaysis can be used to establish the materials properties and the geometric characteristics of an assembly, in which the induced stresses and the bow of the adherends will not exceed the allowable value. The taken approach can be easily generalized for configurations other than an elongated plate.

The adherend strips in an elongated adhesively bonded assembly with identical adherends can be treated, from the standpoint of structural analysis, as thin long-and-narrow plates lying on a continuous elastic foundation provided by the adhesive layer (Fig. 4). If the assembly is manufactured at an elevated temperature and is subsequently cooled down to a low temperature, then the interfacial shearing, $\tau(x)$, and 'peeling', $p(x)$, stresses arise because of the thermal contraction mismatch of the adhesive material with the material of the adherends. These stresses result in the bow of the adherend plates (strips). The elastic curve, $w(x)$, of an adherend, subjected to the action of the interfacial stresses, can be found from the equation:

$$Dw''(x) = \int_{-l}^{x}\int_{-l}^{\xi'} p(\xi)\,\mathrm{d}\xi\,\mathrm{d}\xi' - T(x)\frac{h}{2}. \tag{A1}$$

Here, $w(x)$ is the deflection function, $D = [Eh^3/(12(1-\nu)^2)]$ is the adherend's flexural rigidity, E and ν are the elastic constants of the adherends material, h is the adherend thickness, l is half the assembly length,

$$T(x) = \int_{-l}^{x} \tau(\xi)\,\mathrm{d}\xi \tag{A2}$$

is the force acting in the given cross-section of the adherend, x, and $\tau(x)$ and $p(x)$ are the interfacial shearing stress and the through-thickness ('peeling') stress, respectively. The origin, O, of the coordinate x is in the mid-cross-section of the adherend plate.

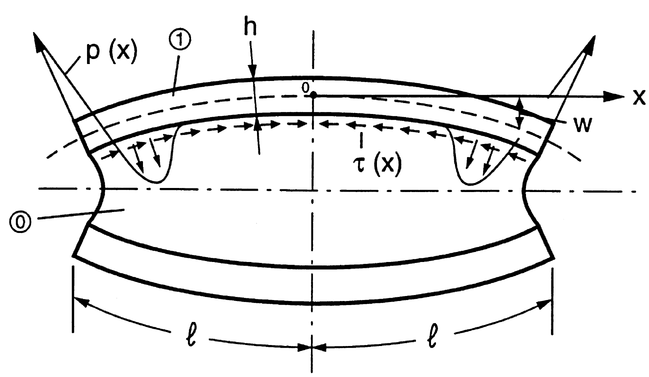

Fig. 4. An adhesively bonded assembly with identical adherends subjected to bending.

Because of the symmetry of the deformations of the assembly with respect to its midplane, the 'peeling' stress, $p(x)$, can be sought, in an approximate analysis, in the form:

$$p(x) = -2Kw(x), \tag{A3}$$

where

$$K \simeq \frac{E_0}{(1 - \nu_0^2)h_0}$$

is the spring constant of the elastic foundation, provided by the adhesive layer, E_0 and ν_0 are the elastic constants of the adhesive material, and h_o is the thickness of the adhesive layer.

Differentiating eqn (A1) twice, and considering the relationships (A2) and (A3), we obtain the following equation for the unknown interfacial stresses, $\tau(x)$ and $p(x)$:

$$p^{\mathrm{IV}}(x) + 4\alpha^4 p(x) = 2h\alpha^4\tau'(x), \tag{A4}$$

where

$$\alpha = \left(\frac{K}{2D}\right)^{1/4}. \tag{A5}$$

In order to obtain an additional relationship for the functions $\tau(x)$ and $p(x)$, we use the condition

$$u_0(x) = u_1(x) \tag{A6}$$

of the compatibility of the longitudinal interfacial displacements, $u_0(x)$ and $u_1(x)$, of the adhesive layer, and one of the adherend plates. If the stresses $\tau(x)$ were known, then these displacements could be evaluated by the approximate formulas (Suhir, 1986):

$$u_0(x) = -\alpha_0 \Delta t x - \lambda_0 \int_0^x T(\xi)\,\mathrm{d}\xi + \kappa_0\tau(x)$$

$$u_1(x) = -\alpha_1 \Delta t x + 2\lambda_1 \int_0^x T(\xi)\,\mathrm{d}\xi - 2\kappa_1\tau(x), \tag{A7}$$

where α_0 and α_1 are the coefficients of thermal expansion (contraction) of the adhesive and the adherend materials, respectively,

$$\lambda_0 = \frac{1 - \nu_0}{E_0 h_0}, \quad \lambda_1 = \frac{1 - \nu_1}{E_1 h_1}$$

are the longitudinal compliances of the adhesive layer and one of the adherends,

$$\kappa_0 = \frac{h_0}{6G_0} = \frac{1}{3}(1 + \nu_0)\frac{h_0}{E_0}$$

$$\kappa_1 = \frac{h_1}{3G_1} = \frac{2}{3}(1 + \nu_1)\frac{h_1}{E_1}$$

are the interfacial compliances of the adhesive layer and the adherend plates, respectively, and Δt is the change in temperature.

Introducing the formulas (A7) into the displacement compatibility condition (A6) and accounting for the fact that the compliances, λ_1 and κ_1, of the adherends are significantly smaller than the compliances, λ_0 and κ_0, of the adhesive, we obtain the following integral equation for the interfacial shearing stress function, $\tau(x)$:

$$\lambda_0 \int_0^x T(\xi)\,\mathrm{d}\xi - \kappa_0 \tau(x) = -\Delta\alpha\Delta t x, \tag{A8}$$

where $\Delta\alpha = \alpha_0 - \alpha_1$ is the difference in the coefficients of expansion of the adhesive and the adherend materials, and the force $T(x)$ is expressed by the formula (A2).

Differentiating eqn (A8) with respect to the coordinate x, we obtain:

$$\lambda_0 T(x) - \kappa_0 \tau'(x) = -\Delta\alpha\Delta t. \tag{A9}$$

Since $T(l) = 0$ (there are no external longitudinal forces acting at the edge $x = l$), eqn (A9) results in the following boundary condition for the interfacial shearing stress function $\tau(x)$:

$$\tau'(l) = -\frac{\Delta\alpha\Delta t}{\kappa_0}. \tag{A10}$$

Differentiating eqn (A9) and considering the relationship (A2), we obtain the following differential equation for the shearing stress function, $\tau(x)$:

$$\tau''(x) - k^2 \tau(x) = 0, \tag{A11}$$

where

$$k = \sqrt{\frac{\lambda_0}{\kappa_0}} = \frac{1}{h_0}\sqrt{3\frac{1 - \nu_0}{1 + \nu_0}}.$$

is the factor of the interfacial stress.

Interfacial stresses

The eqn (A11) has the following solution:

$$\tau(x) = C_0 \sinh kx + C_1 \cosh kx, \tag{A12}$$

where C_0 and C_1 are constants of integration. The interfacial shearing stress, $\tau(x)$, must be antisymmetric with respect to the origin. Therefore one should put $C_1 = 0$, and the solution (A12) yields:

$$\tau(x) = C_0 \sinh kx. \tag{A13}$$

Introducing this formula into the condition (A10), we obtain:

$$C_0 = \frac{\Delta\alpha\Delta t}{k\kappa_0} \frac{1}{\cosh kl}.$$

With this constant of integration, the formula (A13) results in the following expression for the interfacial shearing stress:

$$\tau(x) = \frac{\Delta\alpha\Delta t}{k\kappa_0}\frac{\sinh kx}{\cosh kl} = k\frac{\Delta\alpha\Delta t}{\lambda_0}\frac{\sinh kx}{\cosh kl} = 2G_0\sqrt{3\frac{1+v_0}{1-v_0}}\Delta\alpha\Delta t\frac{\sinh kx}{\cosh kl}, \tag{A14}$$

where $G_0 = [E_0/(2(1+v_0))]$ is the shear modulus of the adhesive material.

The formula (A14) indicates that in a situation, when Young's modulus of the adhesive material is significantly lower than Young's modulus of the adherends, the thickness of the adhesive layer does not affect the interfacial shearing stress. This formula indicates also that the interfacial shearing stress is proportional to the shear modulus of the adhesive material.

With the formula (A14) for the shearing stress, eqn (A4) for the peeling stress can be written as follows:

$$p^{\mathrm{IV}}(x) + 4\alpha^4 p(x) = 2h\alpha^4\frac{\Delta\alpha\Delta t}{\kappa_0}\frac{\cosh kx}{\cosh kl}. \tag{A15}$$

This equation is not different from the well-known equation of bending of a beam lying on a continuous elastic foundation (see, for instance, Timoshenko, 1956). Therefore the solution to eqn (A15) can be sought in the form (see, for instance, Suhir, 1991):

$$p(x) = D_0V_0(\alpha x) + D_2V_2(\alpha x) - A\frac{\cosh kx}{\cosh kl}, \tag{A16}$$

where D_0 and D_2 are constants of integration. The first two terms in this expression provide the general solution to the homogeneous equation corresponding to the nonhomogeneous eqn (A15). The third term is the particular solution to the eqn (A15). The constant A in this solution can be found as

$$A = -\frac{2h\alpha^4\Delta\alpha\Delta t}{\kappa_0(k^4+4\alpha^4)}.$$

The functions $V_0(\alpha x)$ and $V_2(\alpha x)$ in (A16) are expressed by the formulas

$$V_0(\alpha x) = \cosh \alpha x \cos \alpha x, \quad V_2(\alpha x) = \sinh \alpha x \sin \alpha x,$$

and obey the following rules of differentiation:

$$V_0'(\alpha x) = -\alpha\sqrt{2}V_3(\alpha x), \quad V_1'(\alpha x) = \alpha\sqrt{2}V_0\ (\alpha x),$$

$$V_2'(\alpha x) = \alpha\sqrt{2}V_1(\alpha x), \quad V_3'(\alpha x) = \alpha\sqrt{2}V_2\ (\alpha x).$$

The functions $V_1(\alpha x)$ and $V_2(\alpha x)$ are expressed as

$$V_1(\alpha x) = \frac{1}{\sqrt{2}}(\cosh \alpha x \sin \alpha x + \sinh \alpha x \cos \alpha x),$$

$$V_3(\alpha x) = \frac{1}{\sqrt{2}}(\cosh \alpha x \sin \alpha x - \sinh \alpha x \cos \alpha x).$$

The functions $V_i(\alpha x)$, $i = 0, 1, 2, 3$, are tabulated (Suhir, 1991).

The deflection function, $w(x)$, of the adherend must satisfy the boundary conditions:

$$w''(l) = 0, \quad w'''(l) = 0. \tag{A17}$$

These conditions indicate that there are no external concentrated bending moments, nor concentrated lateral forces acting at the end $x = l$. The conditions (A17) lead, with consideration of the relationship (A3), to the following boundary conditions for the peeling stress function, $p(x)$:

$$p''(l) = 0, \quad p'''(l) = 0. \tag{A18}$$

Introducing (A16) into these conditions, we obtain the following algebraic equations for the constants D_0 and D_2:

$$V_2(u)D_0 - V_0(u)D_2 = -\frac{k^2 A}{2\alpha^2}$$

$$V_1(u)D_0 + V_3(u)D_2 = -\frac{k^3 A}{2\sqrt{2}\alpha^3}\tanh kl, \tag{A19}$$

where the parameter u is expressed as

$$u = \alpha l = l\left(\frac{K}{2D}\right)^{1/4}.$$

From eqns (A19) we find:

$$D_0 = -\frac{k^2 A}{2\sqrt{2}\alpha^3}\frac{kV_0(u)\tanh kl + \alpha\sqrt{2}V_3\ (u)}{V_0(u)V_1(u) + V_2(u)V_3\ (u)}$$

$$= -\frac{k^2 A}{\alpha^3}\frac{kV_0(u)\tanh kl + \alpha\sqrt{2}V_3\ (u)}{\sinh 2u + \sin 2u}$$

$$D_2 = -\frac{k^2 A}{2\sqrt{2}\alpha^3}\frac{kV_2(u)\tanh kl - \alpha\sqrt{2}V_1\ (u)}{V_0(u)V_1(u) + V_2(u)V_3\ (u)}$$

$$= -\frac{k^2 A}{\alpha^3}\frac{kV_2(u)\tanh kl + \alpha\sqrt{2}V_1\ (u)}{\sinh 2u + \sin 2u}.$$

Lateral load

The total lateral ('through-thickness') load, $q(x)$, can be found as the sum of the 'peeling' load, $p(x)$, expressed by eqn (A16), and the additional load, $-(h/2)\tau'(x)$, which is due to the nonuniform longitudinal distribution of the interfacial shearing stress:

$$q(x) = p(x) - \frac{h}{2}\tau'(x) = D_0 V_0(\alpha x) + D_2 V_2(\alpha x) + \frac{k^4 A}{4\alpha^4}\frac{\cosh kx}{\cosh kl}.$$

This expression can be obtained also directly from eqn (A1).

The lateral load, $q(x)$, is self-equilibrated. Indeed,

$$\int_0^l q(x)\,dx = \frac{1}{\alpha\sqrt{2}}\left[D_0\ V_1(u) + D_2\ V_3(u)\right] + \frac{k^3 A}{4\alpha^4}\tanh kl$$

$$= \frac{1}{\alpha\sqrt{2}}\frac{k^3 A}{2\sqrt{2}\alpha^3}\tanh kl + \frac{k^3 A}{4\alpha^4}\tanh kl = 0, \tag{A20}$$

and

$$\int_0^l \int_0^x q(\xi)\,\mathrm{d}\xi\,\mathrm{d}\xi' = \frac{1}{2\alpha^2}\left[D_0\ V_2(u) - D_2\ V_0(u)\right] + \frac{k^2 A}{4\alpha^4}$$

$$= \frac{1}{2\alpha^2}\frac{k^2 A}{2\alpha^2} + \frac{k^2 A}{4\alpha^4} = 0.$$

Lateral forces and bending moments in the adherends

The lateral force, $N(x)$, in the adherends can be evaluated as

$$N(x) = \int_0^x q(x)\,\mathrm{d}x = \frac{1}{\alpha\sqrt{2}}\left[D_0\ V_1(\alpha x) + D_2\ V_3\ (\alpha x)\right] + \frac{k^3 A}{4\alpha^4}\frac{\sinh kx}{\cosh kl}. \tag{A21}$$

This force is zero at the origin ($x = 0$). In addition, as evident from (A20), the lateral force is equal to zero at the end $x = l$. The formula (A21) indicates that the lateral force function, $N(x)$, is antisymmetric with respect to the origin: $N(x) = -N(-x)$.

The bending moment can be obtained from (A21) by integration:

$$M(x) = \int_{-l}^x N(\xi)\,\mathrm{d}\xi = \frac{D_0}{2\alpha^2}\left[V_2\ (\alpha x) - V_2(u)\right] - \frac{D_2}{2\alpha^2}\left[V_0\ (\alpha x) - V_0(u)\right] - \frac{k^2 A}{4\alpha^4}\left(1 - \frac{\cosh kx}{\cosh kl}\right). \tag{A22}$$

The moment $M(x)$ is symmetric with respect to the origin: $M(x) = M(-x)$. This moment is equal to zero at the end $x = l$.

Deflection function

The deflection function, $w(x)$, of the adherend can be found from the equilibrium equation (equation of bending)

$$Dw''(x) = M(x).$$

Using (A22), we obtain:

$$Dw'(x) = -\frac{D_0}{2\alpha^2}\left[\frac{1}{\alpha\sqrt{2}}V_3\ (\alpha x) - xV_2\ (u)\right] + \frac{D_2}{2\alpha^2}\left[\frac{1}{\alpha\sqrt{2}}V_1\ (\alpha x) - xV_0\ (u)\right] + \frac{k^2 A}{4\alpha^4}\left(x - \frac{1}{k}\frac{\sinh kx}{\cosh kl}\right).$$

The constant of integration is put equal to zero in this equation, since the deflection function, $w(x)$, must be symmetric with respect to the origin, and therefore the angle of rotation at the origin must be zero: $w'(0) = 0$. The integration of the above equation yields:

$$Dw(x) = \frac{D_0}{2\alpha^2}\left[\frac{1}{2\alpha^2}V_0\ (\alpha x) + \frac{1}{2}x^2 V_2\ (u)\right] + \frac{D_2}{2\alpha^2}\left[\frac{1}{2\alpha^2}V_2\ (\alpha x) - \frac{1}{2}x^2 V_0\ (u)\right]$$

$$+\frac{k^2 A}{4\alpha^4}\left(\frac{1}{2}x^2 - \frac{1}{k^2}\frac{\cosh kx}{\cosh kl}\right) + C$$

$$= \frac{1}{4\alpha^4}\left[D_0 V_0\ (\alpha x) + D_2\ V_2\ (\alpha x) - A\frac{\cosh kx}{\cosh kl}\right]$$

$$+\frac{x^2}{4\alpha^2}\left(D_0 V_2(u) - D_2\ V_0(u) + \frac{k^2 A}{2\alpha^2}\right) + C.$$

As evident from the first formula in (A19), the expression in parentheses in this equation is equal to zero, and therefore,

$$Dw(x) = \frac{1}{4\alpha^4}\left[D_0 V_0\ (\alpha x) + D_2\ V_2\ (\alpha x) - A\frac{\cosh kx}{\cosh kl}\right] + C. \tag{A23}$$

The constant C in the obtained equation can be chosen in an arbitrary fashion. If, for instance, $C = 0$, then, considering (A5), we have:

$$w(x) = \frac{1}{2K}\left[D_0 V_0(\alpha x) + D_2 V_2(\alpha x) - A\frac{\cosh kx}{\cosh kl}\right]$$

$$= \frac{1}{2K}\left[q(x) - A\left(1 + \frac{k^4}{4\alpha^4}\right)\frac{\cosh kx}{\cosh kl}\right]. \tag{A24}$$

This formula can be used to compute the ordinates of the deflection curve of the adherend. It can be used also to design an assembly, in which the adherends bow is as small, as necessary, whether mechanical or any other (say, optical) criteria are considered.

Another way of calculating the ordinates, $w(x)$, of the deflection curve can be based on the direct numerical integration of the lateral load, $q(x)$. If this approach is used, one should evaluate first the lateral force

$$N(x) = \int_0^x q(\xi)\,\mathrm{d}\xi,$$

then compute the bending moment

$$M(x) = \int_0^x N(\xi)\,\mathrm{d}\xi + \int_0^l q(\xi)\xi\,\mathrm{d}\xi,$$

and, finally, calculate the deflection curve by the formula:

$$w(x) = \frac{1}{D}\int_0^x\int_0^x M(\xi)\,\mathrm{d}\xi\,\mathrm{d}\xi' + C.$$

This formula produces the same results as the formula (A24).

Numerical data

The numerical data were obtained for a 100 mm long assembly ($l = 50$ mm) with 1 mm thick glass adherends ($E_1 = 7384$ kg/mm^2, $v_1 = 0.20$, $\alpha_1 = 0.5 \times 10^{-6} 1/°\mathrm{C}$). Two types of adhesive material were considered: a silicone gel type ($E_0 = 200$ psi $= 0.141 kg/\mathrm{mm}^2$, $v_0 = 0.490$, $\alpha_0 = 200 \times 10^{-6} 1/°\mathrm{C}$) and an epoxy type $E_0 = 100{,}000$ psi $= 70.323 kg/\mathrm{mm}^2$, $v_0 = 0.420$, $\alpha_0 = 70 \times 10^{-6} 1/°\mathrm{C}$). The calculations were performed for 0.25, 1.00, 2.00 and 5.00 mm thick adhesive layers. The assumed change in temperature was $\Delta t = 150°\mathrm{C}$.

The distribution of the interfacial shearing stresses along the assembly is shown in Fig. 5, for the silicone gel type adhesive material, and in Fig. 11, for the epoxy type adhesive. The 'peeling' stresses are shown in Figs. 6 and 12 for the silicone gel adhesive and epoxy adhesive, respectively. The total lateral load, which is due to the peeling stress and the additional component caused by the nonuniform distribution of the interfacial shearing stress, is shown in Figs. 7 and 13 for the two types of the adhesives considered. The lateral (shearing) forces in the adherends are shown in Figs. 8 and 14, for the cases of a silicone-gel type of adhesive and for an epoxy type of adhesive, respectively. The bending moments are shown in Figs. 9 and 15. Finally, the reflections of the adherends are plotted in Figs. 10 and 16.

As evident from the obtained results, the application of a high-modulus epoxy adhesive (in comparison with the gel) results in substantially higher stresses than in an assembly with the silicone gel adhesive, despite the significantly higher coefficient of expansion (contraction) of the silicone gel material. As to the maximum deflections, these are quite comparable: the effect of the high Young's modulus of the epoxy is 'outweighed' to a great extent by its low coefficient of expansion. The calculated data indicate that thicker adhesives result in lower stresses in, and in higher displacements of, the adherend strips.

The interfacial stresses responsible for the would-be adhesive failure of the adhesive material are significantly higher in the case of the epoxy adhesive than in the case of the silicone gel adhesive. It does not mean, however, that the epoxy should be definitely regarded less preferable: its ultimate adhesive strength can be significantly higher than the adhesive strength of the silicone gel, and therefore the integrity of an epoxy bonded assembly might not be compromised despite the high level of the induced interfacial stresses.

The bending moments, responsible for the strength and the bow of the adherends, are substantially larger in the case of an epoxy bonded assembly than in the case of the silicone gel. Note that the maximum bending moments occur at the cross-sections which are close to the ends of the assembly. The thickness of the adhesive layer has a relatively small effect on the maximum bending moment in an assembly bonded with silicone gel, but has an appreciable effect on the maximum bending moment in the case of an epoxy bonded assembly: the maximum bending moment decreases with an increase in the thickness of this layer.

Summary

The obtained results can be used to establish the size of the midportion of the assembly within which the thermally induced deflections of the adherends are sufficiently low. In the carried out examples, such a 'midportion' occupies, in the case of an epoxy bonded assembly, about 80% of the assembly length with a 0.25 mm thick epoxy and only about 50% of the assembly length with a 5 mm thick epoxy. In the case of a silicone gel adhesive, however, there is practically no 'deflection free' midportion of the assembly. The situation can be improved, if necessary, by using thicker adherends and thinner adhesive layers, and/or by employing adherends with a better thermal match with the adhesive.

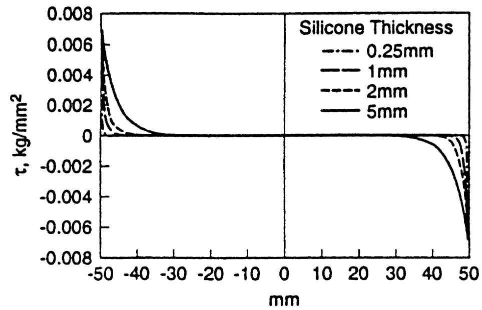

Fig. 5. Shearing stress.

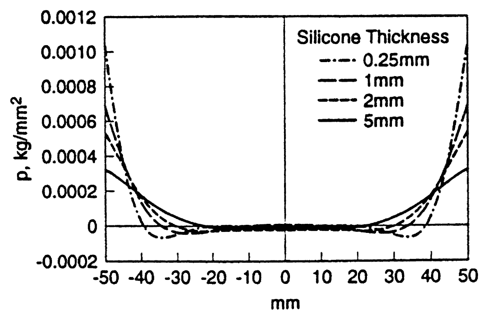

Fig. 6. Peeling stress.

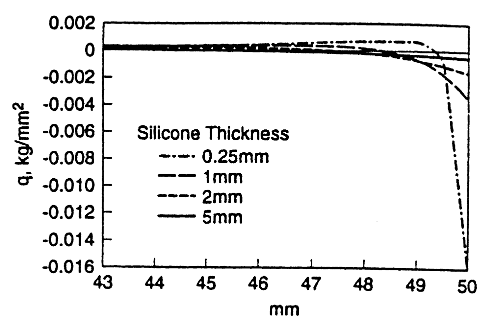

Fig. 7. Lateral load.

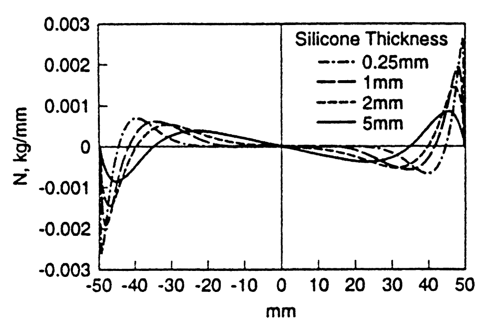

Fig. 8. Lateral forces.

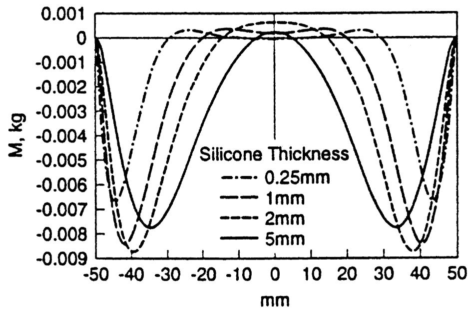

Fig. 9. Bending moments.

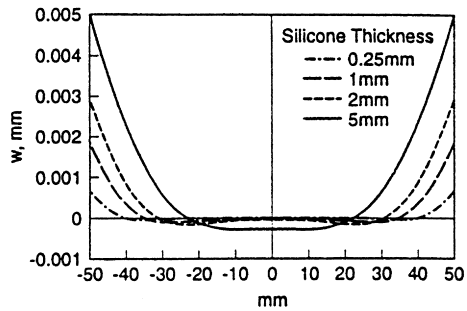

Fig. 10. Deflections.

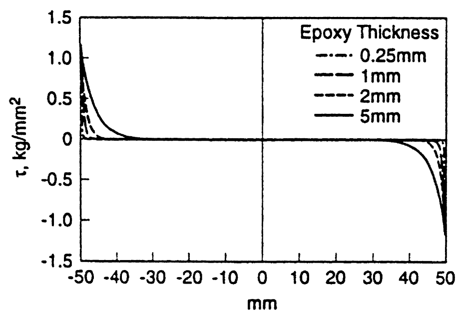

Fig. 11. Shearing stress.

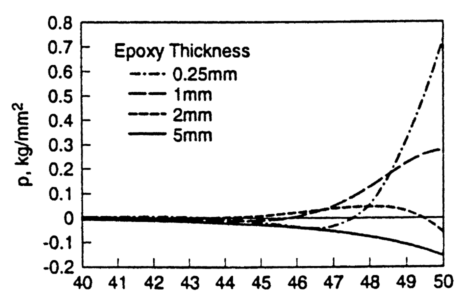

Fig. 12. Peeling stress.

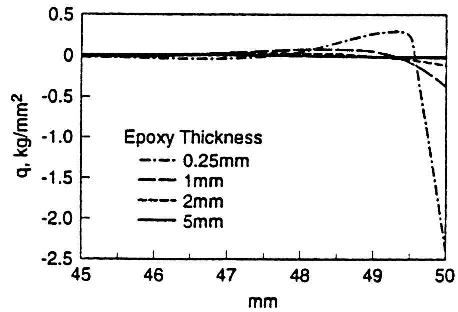

Fig. 13. Lateral load.

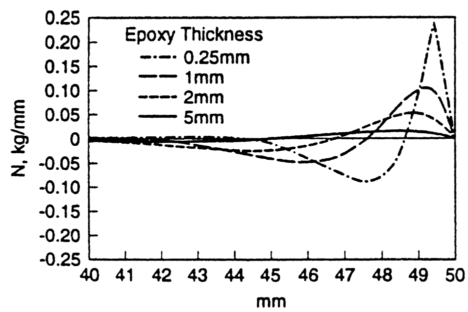

Fig. 14. Lateral forces.

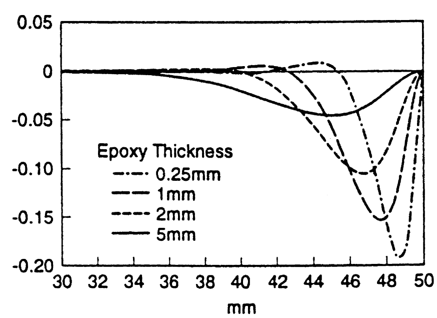

Fig. 15. Bending moments.

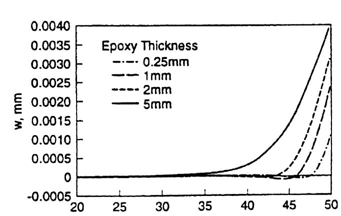

Fig. 16. Deflections.

Appendix B. Adhesively bonded assembly subjected to shear

Let an adhesively bonded assembly with identical nondeformable adherends and a homogeneous adhesive layer be subjected to an external force, $\hat{T}$, applied to the edge of one of the adherends (Fig. 17). The second adherend is assumed to be fixed. The induced force, $T(x)$, acting at an inner cross-section, x, of the adhesive layer, can be sought in the form:

$$T(x) = C\cosh kx + D\sinh kx, \tag{B1}$$

where C and D are constants of integration,

$$k = \sqrt{\frac{\lambda}{\kappa}}$$

is the factor of the shearing stress,

$$\lambda = \frac{1}{E^0 h_0}, \tag{B2}$$

is the longitudinal axial compliance of the adhesive layer, h_0 is the thickness of this layer, $E^0 = [E/(1-v)]$, is the generalized Young's modulus, E is Young's modulus of the adhesive material, v is its Poisson's ratio,

$$\kappa = \frac{1}{E}\frac{\sum_{i=1}^{\infty}\gamma_i K(u_i)\sin\alpha_i x}{\sum_{i=1}^{\infty}\gamma_i\alpha_i\sin\alpha_i x}, \quad \alpha_i = \frac{i\pi}{2l}, \quad i = 1, 3, 5, \ldots \tag{B3}$$

is the longitudinal interfacial (shearing) compliance of the adhesive layer (Suhir, 1986),

$$u_i = \alpha_i\frac{h_0}{2} = \frac{i\pi}{4}\frac{h_0}{l}, \quad i = 1, 3, 5, \ldots$$

is the parameter of the thickness-to-length ratio for this layer, and

$$\gamma_i = \frac{2}{\alpha_i l}\int_0^l \tau(x)\sin\alpha_i x\,\mathrm{d}x, \quad i = 1, 3, 5, \ldots$$

The function $K(u_1)$ in the formula (B3) is expressed as follows (Suhir, 1986):

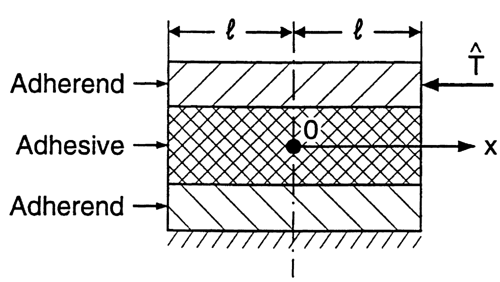

Fig. 17. An adhesively bonded joint subjected to shear.

$$K(u_i) = \frac{1+v}{2}\left\{\left[1 - v - (1+v)u_i \operatorname{cotanh} u_i\right]\operatorname{cotanh} u_i + (1+v)u_i + 2\operatorname{cotanh} u_i - \frac{2(1-v)}{u_i}\right\}.$$

The origin, O, of the coordinate x is in the mid-cross-section of the adhesive layer in its midplane.

Introducing (B1) into the boundary conditions

$$T(-l) = 0, \quad T(l) = \hat{T},$$

for the induced force, $T(x)$, we obtain the following formulae for the constants of integration:

$$C = \frac{\hat{T}}{2\cosh kl}, \quad D = \frac{\hat{T}}{2\sinh kl}. \tag{B4}$$

The formula for the shearing stress can be obtained from (B1) by differentiation:

$$\tau(x) = \frac{\mathrm{d}T(x)}{\mathrm{d}x} = k(C\sinh kx + D\cosh kx).$$

The shearing stress at the edges of the assembly can be found, using the formulas (B4), as follows:

$$\tau(-l) = \frac{k\hat{T}}{\sinh 2kl}, \quad \tau(l) = k\hat{T}\operatorname{cotanh} 2kl.$$

The displacements at the assembly edges can be evaluated by the formulas:

$$u(-l) = \kappa\tau(-l) = \frac{k\kappa\hat{T}}{\sinh 2kl}, \quad u(l) = \kappa\tau(l) = k\kappa\hat{T}\operatorname{cotanh} 2kl. \tag{B5}$$

For sufficiently long assemblies ($kl > 2.5$), the displacement, $u(-l)$, at the edge $x = -l$ is zero, and the displacement, $u(l)$, at the edge $x = l$ is length independent:

$$u(l) = k\kappa\hat{T}.$$

For very short assemblies ($kl < 0.1$), both the edge displacements are approximately the same:

$$u(-l) \simeq u(l) = \frac{\kappa\hat{T}}{2l}.$$

As one can see from this formula, the displacements at the assembly edges are inversely proportional to its length.

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